

Two generalizations of the Boltzmann equation

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Received 28 September 2005 / Received in final form 23 November 2005

Published online 12 April 2006 – © EDP Sciences, Società Italiana di Fisica, Springer-Verlag 2006

Abstract. We connect two different extensions of Boltzmann's kinetic theory by requiring the same stationary solution. Non-extensive statistics can be produced by either using corresponding collision rates nonlinear in the one-particle densities or equivalently by using nontrivial energy composition rules in the energy conservation constraint part. Direct transformation formulas between key functions of the two approaches are given.

PACS. 25.75.Nq Quark deconfinement, quark-gluon plasma production, and phase transitions – 05.20.Dd Kinetic theory – 05.90.+m Other topics in statistical physics, thermodynamics, and nonlinear dynamical systems (restricted to new topics in section 05) – 02.70.Ns Molecular dynamics and particle methods

Power-law distributions in nature are nearly as common as Gaussian distributions, which is the limiting distribution emerging for the properly scaled sum of infinitely many independent random variables. The power-law tailed distributions have escaped somehow the strength of the central limit theorem: either by being composed from long-tailed individual distributions, or by featuring finite-size effects in a general sense. To the latter category belong unscreened long-range forces and long-time memory effects; to the former a (multi)fractal phase space occupation. Since power-law tailed distributions can be found in many areas from particle physics and astrophysics to financial market models [1–3], it is righteously suspected that these belong to some more or less universal stationary states of complex dynamics. The question arises whether there are characteristic common features in these dynamics and what they then are.

Non-extensive thermodynamics has been developed in the past two decades as a statistical theory to deal with such physical stationary states [4–6]. Initially based on mathematical investigations of a generalized definition of Boltzmann's entropy, a never-decreasing macroscopic state-parameter intimately connected to statistical probabilities of microstates, this theory soon started to study dynamics possibly leading to such states. Studies of anomalous diffusion, random walk, and noisy equations [7–10] revealed that either a nonlinearity in the one-particle probability or a nontrivial interaction with the environment, which depends on the observed low degree

of freedom subsystem itself, may be responsible for such a nontraditional behavior.

While the above approaches all mounded in the study of an extended Fokker-Planck problem, the set of all dynamical evolutions leading to a thermodynamical state is much wider. In general a considerable fluctuation of the intensive parameters may establish a non-Boltzmannian canonical distribution [11]. Another classical field of establishing thermodynamics (and hence also non-extensive thermodynamics) is kinetic theory in general. The basic assumption of Boltzmann, that the time evolution of many systems has a fast, micro-reversible and ergodizing component, which he comprised into a collision integral, opens up studies of more complex non-linear dynamical theories.

Boltzmann's original kinetic theory is based upon a collision rate, which is multilinear in the one-particle densities and symmetric both in the colliding partners and against time reversal. Based on these properties the H-theorem was derived, funding the quantity entropy, and giving a way towards a microscopic establishment of thermodynamics. The Boltzmann entropy is an extensive (additive) quantity reflecting the independence of microstate probabilities in weakly interacting subsystems. The classical Boltzmann equation restricts possible stationary distributions to satisfy a product rule, once an addition rule for energy is given.

Recent generalizations of the Boltzmann equation hook in exactly at this point: either the product formula is generalized [12,13], interpretable as mimicking nontrivial many-body correlations in the equilibrium state, or the addition of energy is replaced by a more general formula [14], accounting for an in-medium interaction energy shared by

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the colliding partners. These two approaches may lead to the same, non-standard stationary one-particle distribution. The aim of this work is to clarify the interrelations between them and to present corresponding mathematical formulas.

In reference [12] possible extensions with nonlinear dependence of the collision rate on the one-particle densities have been considered, this approach we shall refer here as the nonlinear Boltzmann equation (NLBE). The rate in this approach consists of production and blocking factors, both in gain and loss terms. Considering a simple $1+2 \leftrightarrow 3+4$ two to two body collision, the rate of change of the one-particle phase space density, $f_1 = f(\mathbf{p}_1)$, is given by

$$\dot{f}_1 = \int_{234} w_{1234} (a_3 b_1 a_4 b_2 - a_1 b_3 a_2 b_4) \quad (1)$$

with $a_i = a(f_i)$ being the general production and $b_i = b(f_i)$ the blocking factors. The transition probability rate factor, w_{1234} may contain a 1–3 and a 2–4 symmetric contribution also from correlations between production and blocking. This is a general extension of the Boltzmann and the Boltzmann Uhling-Uehlenbeck equations with respect to nonlinearity of the collision rate. The standard theories are recovered for $a(f) = f$ and $b(f) = 1$, or $b(f) = 1 \pm f$ respectively. In this case the stationary distribution is governed by the ratio $\kappa(f) = a(f)/b(f)$ which becomes the traditional Boltzmann factor:

$$\kappa(f_{eq}) = \exp(-E/T). \quad (2)$$

This result assumes that in two-body collisions momenta and energy are composed additively ($E_1 + E_2 = E_3 + E_4$, $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4$). An H-theorem can be proven with the generalized expression for the entropy,

$$S_K = \int_1 \sigma(f_1) \quad (3)$$

where the integration is over the one-particle phase space. It turns out that $\sigma(f)$ is related to the previous quantities via

$$\sigma'(f) = -\ln \kappa(f). \quad (4)$$

Recently another approach to generalize Boltzmann's original treatment has been proposed [14]. Here the multilinearity of the collision rate is kept, blocking factors are not applied, but the additivity of the energy during the micro-collisions is replaced by a more general requirement: only a given function of the individual energies, physically standing for the total two-particle energy, is conserved, $h(E_1, E_2) = h(E_3, E_4)$. The function $h(x, y)$ describes a general, non-extensive energy composition rule for the two-body system. If this is chosen with the property of associativity, $h(h(x, y), z) = h(x, h(y, z))$, then its most general form is related to a strict monotonic function, $X(x)$:

$$h(x, y) = X^{-1}(X(x) + X(y)). \quad (5)$$

The function $X(x)$ is a mapping of the non-extensive composition rule to the addition rule, it is unique up to a real

multiplicative factor. The stationary distribution in this case is given by

$$f_{eq} = \exp(-X(E)/T). \quad (6)$$

The H-theorem can be proven for Boltzmann's original construction, but this quantity may be interpreted as the total of the additive mappings of non-extensive entropy contributions:

$$S_B = X_s(S_N) = -\int f \ln f. \quad (7)$$

By requiring a connection to the stationary state described by the previous approach, it turns out that the same composition rule should be applied to the energy (scaled by the temperature) and to the entropy, i.e. $X(E)/T = X_s(E/T)$ with the respective mapping function for energy and entropy. The non-extensive Boltzmann equation (NEBE) approach is still multilinear in the one-particle distributions. In parton cascade simulations, by applying the above rules, the non-extensively composed total energy is conserved $E_{tot} = X^{-1}(\sum_i X(E_i))$.

Of course, one may consider to apply both extensions, the nonlinear density dependence and the non-extensive energy composition rule. In order to support a given non-extensive thermodynamics, however, each one alone suffices. In this respect these two generalizations are equivalent and transformation formulas can be obtained between them. In the followings we review this correspondence.

A common stationary distribution relates the $\kappa(f)$ and the $X_s(E/T)$ functions:

$$f_{eq} = \kappa^{-1}(e^{-x}) = e^{-X_s(x)} \quad (8)$$

with the argument $x = E/T$. This allows for obtaining both $\kappa(f)$ or $X_s(E/T)$ to a given distribution, or converting these two basic functions into each other. Based on this the non-extensive entropy formulas (3) and (7) can be derived, the relation

$$\sigma'(f) = -\ln \kappa(f) = X_s^{-1}(-\ln f) \quad (9)$$

is just a consequence thereof. Applying this relation to the Boltzmann entropy (7) we arrive at

$$X_s(S_N) = \int f X_s(s(f)), \quad (10)$$

with the one-particle contribution $s(f) = \sigma'(f)$ at the generalization of the mapping of the non-extensive entropy to an additive entropy measure, S_B . The basic entropy transformers of the two approaches, $\sigma(f)$ and $X_s(E/T)$ must be related by equation (9) in order to describe the same non-extensive thermodynamics in the stationary state. As a trivial consequence the same composition rule, $h(x, y)$, is applied to the energy and to the entropy up to a scale factor T .

It is noteworthy that the Tsallis entropy can actually be obtained by considering $S_T = \int f s(f)$. This very expression fulfills an H-theorem in the case of Tsallis distribution, the corresponding Tsallis and NLBE $\sigma'(f)$ -s

being linear functions of each other (see below). In the general case they are, however, different. The deformed exponential and logarithm functions, considered in several approaches to the non-extensive thermodynamics, are simply related to the scaled mapping function [15]:

$$\exp_{def}(t) = e^{-X_s(-t)}, \quad \ln_{def}(f) = -X_s^{-1}(-\ln f). \quad (11)$$

Finally we demonstrate the above relations for some well known stationary distributions, relating them to specific non-extensive composition rules, production and blocking factors, or expressions for the total entropy, respectively.

The Maxwell-Boltzmann distribution, or in general the Boltzmann-Gibbs distribution, $f(E) = \exp(-E/T)$ is recovered as a particular case in both approaches. There is no blocking factor, $b(f) = 1$, the production factor is linear, $a(f) = f$. The mapping function is the identity, $X_s(t) = t$ and the energy (and entropy) composition rule is the simple addition, $h(x, y) = x + y$.

The Tsallis distribution, $f(E) = (1 + aE)^{-1/aT}$ arises for $X_s(x) = \frac{1}{a} \ln(1 + ax)$ [16] by the energy composition rule $h(x, y) = x + y + axy$ [6]. The production to blocking factor ratio becomes $\kappa(f) = a(f)/b(f) = \exp((1 - f^{-aT})/aT)$. The Tsallis index is given by $q = 1 - aT$. Note that if the parameter a in the generalized composition rule is temperature independent, then the resulting Tsallis index will depend on temperature. The NLBE entropy is given by $S_K = \frac{1}{1-q} \int (f^q/q - f)$, the Tsallis entropy by $S_T = \frac{1}{1-q} \int (f^q - f)$. This is not a principal difference, because whenever an H-theorem is fulfilled with a given $\sigma'(f)$, a linear combination, $A\sigma'(f) + B$ also fulfills the same H-theorem. The choice $A = q$ and $B = 1$ transforms S_K into S_T in this case.

Its mapping to an additive quantity coincides with the Rényi entropy [17], $S_R = X_s(S_T) = \frac{1}{1-q} \ln \int f^q$, when the distribution f is normalized to one.

A variant of the Lévy distribution, with a fractional power, $f(E) = \exp(-(E/T)^v)$, can be achieved by using the production to blocking ratio $\kappa(f) = \exp(-(-\ln f)^{1/v})$ in the NLBE. The mapping to additive quantity is done by $X(x) = x^v$, generating the abstract composition rule: $h(x, y) = (x^v + y^v)^{1/v}$.

A pure power-law distribution, $f(E) = (bE/T)^{\frac{1}{q-1}}$, may be produced by using $\kappa(f) = \exp(-f^{q-1}/b)$ or equivalently by mapping the energy to an additive quantity via $X(x) = \frac{1}{1-q} \ln bx$. The corresponding composition rule is given by $h(x, y) = bxy$. The NLBE entropy becomes $S_K = \int f^q/bq$, the Tsallis entropy is given by $S_T = \int f^q/b$ in this case. For $q = 1$ they coincide.

The so called quon distribution, $f(E) = \frac{1}{e^{E/T} + q}$ (including the Fermi ($q = 1$) and Bose ($q = -1$) distributions as particular cases), is achieved by the production to blocking ratio: $\kappa(f) = \frac{f}{1-af}$ (the choice $a(f) = f$ and $b(f) = 1 - qf$ leads to the Uhling-Uehlenbeck equation). The corresponding mapping to additive quantity is given by $X(x) = \ln(q + e^x) - \ln(q + 1)$ ($X(0) = 0$ is achieved by a subtraction of a finite constant. This does not work for $q = -1$ signaling the divergence due to Bose condensation.).

The generalized addition rule is somewhat complicated: $h(x, y) = x + y + \ln(1 + q(e^{-x} + e^{-y}) + q(q - 1)e^{-(x+y)}) - \ln(q + 1)$ features a ‘‘Pauli potential’’, a pair energy leading to an exact Fermi distribution. In the NEBE approach this can be achieved without considering a third (blocking) particle at each collision. The density for the S_K entropy is given by $\sigma(f) = -f \ln f - (1/q - f) \ln(1 - qf)$.

Another distribution, proposed in Ref. [18] for applying to a kinetic theory of relativistic particles, applies the mapping function $X(x) = \frac{1}{k} \cdot \operatorname{asinh}(kx)$. Another long-known formula, the mapping of relativistic velocities to an additive rapidity variable, follows from Einstein velocity-composition formula: $X(v) = c \cdot \operatorname{atanh}(v/c)$.

Some different distributions may be united in two- (or more) parameter classes. For example for small q but arbitrary large x $X(x) = \frac{1}{aq} ((1 + ax)^q - 1)$ approaches Abe’s logarithmic formula leading to the Tsallis distribution, while for large a comes close to $X(x) = x^v$ leading to the Lévy distribution¹.

Finally non-associative composition rules can be simulated in computerized parton cascade simulations, too. In this case, however, no mapping can be found to an additive, statistical quasi-energy. The individual energies after a micro-collision, although random in a certain kinematical range, are no more one-variable functions of the respective energies of the incoming pair, but depend on both energies. Non-associative composition rules, on the other hand, would not be able to converge in the thermodynamical limit of repeated compositions of compositions. Therefore they can probably be only of pure mathematical interest.

We have carried out numerical simulations so far for the Tsallis case [14]. The nonextensive Boltzmann equation is simulated by $N = 10^5 - 10^6$ test particles having initially random momenta \mathbf{p}_i in a certain range and evolving by pairwise changes according to the rules

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4, \\ X(E(p_1)) + X(E(p_2)) = X(E(p_3)) + X(E(p_4)) \quad (12)$$

with the energy mapping $X(E) = \frac{1}{a} \ln(1 + aE)$. This simulation conserves the quasi energy $X(E_{tot}) = \sum_i X(E(p_i))$. Figure 1 shows pairs of momentum vectors satisfying the above conditions for massless (extremely relativistic) particles, i.e. for $E(p_i) = |\mathbf{p}_i|$. The two-dimensional section of the surface of momenta is an ellipse for $a = 0$ (the traditional Boltzmann case) only, for positive or negative values of this parameter a fourth order curve is drawn. Final, stationary distributions of the one-particle bare energy, $f(E)$ with $E = |\mathbf{p}|$ are then Tsallis distributions. Figure 2 shows examples of these distributions. These are numerical demonstrations of how to achieve a Tsallis distribution from arbitrary initial distributions.

In conclusion we have demonstrated that quite different ansätze to generalize Boltzmann’s classical kinetic theory produce the same stationary distributions of non-extensive thermodynamics. Upon this equivalence there

¹ The corresponding deformed exponentials and logarithms are then also two-parameter families.

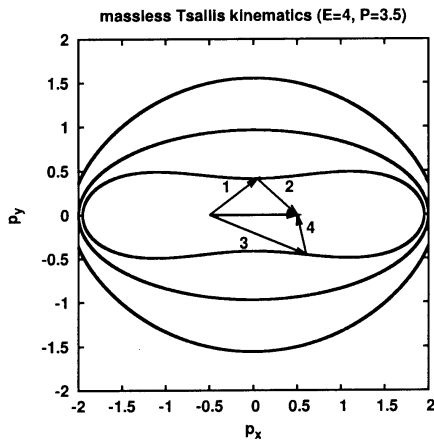


Fig. 1. Kinetic constraint for momentum changes in a pairwise elastic micro-collision using Tsallis rules for the energy composition and massless free dispersion relation. The inner curve corresponds to $a = 0.125$, the middle one to $a = 0$ and the outer one to $a = -0.125$.

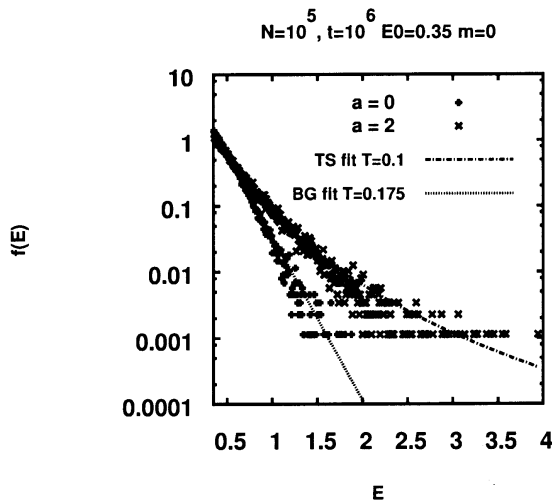


Fig. 2. Tsallis distributions obtained numerically by solving the corresponding nonextensive Boltzmann equation.

always exist a strict monotonous mapping to an additive entropy from more exotic entropy definitions, as long as the energy composition rule applied in the micro-events (collisions) are associative. It is natural to assume that the relevant rules in the thermodynamical limit have to become associative. The H-theorem holds. Several known non-Boltzmann distributions belong to a physically intriguing non-extensive energy addition rule. This gives hope to find realizations in nature by studying the pair interaction mechanism and its thermodynamical limit. Non-linear production and blocking factors may formally be replaced by non-extensive energy formulas if only the stationary distribution is asked for. The second method, not using phase space blocking factors, allows for computer simulations with a resource effort not worse than $\mathcal{O}(N^2)$. Kinetic models leading to non-conventional stationary statistics can be applied to bosonic, fermionic or to quonic systems, to interacting particles with relativistic kinematics, to hadronizing quark matter and in general to nonextensive systems.

Enlightening discussions with Drs. Géza Györgyi at Eötvös University and Antal Jakovác at the Technical University Budapest are hereby gratefully acknowledged. This work has been supported by the Hungarian National Science Fund OTKA (T049466) and the Deutsche Forschungsgemeinschaft.

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